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Matched-asymptotic analysis of low-Reynolds-number flow past two equal circular cylinders

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Jeffery's solution in bi-polar co-ordinates of the two-dimensional Stokes equations cannot be applied to the low-Reynolds-number flow past two parallel circular cylinders because of severe mathematical difficulties. These difficulties can be overcome by considering the flow field far from the cylinders and then modifying the solution near the cylinders so that it becomes the inner expansion for an application of the method of matched asymptotic expansions. After the calculation of the drag, lift and moment coefficients of two adjacent equal circular cylinders to $O(1)$ in the Reynolds number R, the analysis is extended to incorporate partially the effects of fluid inertia of order R. The results show fairly good agreement with Taneda's experimental data.

1. Introduction

In this paper we study two-dimensional low-Reynolds-number flow past two parallel circular cylinders. This has previously been analysed on the basis of Oseen's approximate equations – linearized versions of the Navier–Stokes equations (Fujikawa 1956, 1957; Kuwabara 1957; Yano & Kieda 1980). The analyses were made using polar co-ordinates and are actually perturbation solutions which are valid only for the case of two circular cylinders a great distance apart. A more appropriate approach to this problem is to adopt the matched-asymptotic-expansion method due to Kaplun (1957) and Proudman & Pearson (1957), developed mainly through studies of the low-Reynolds-number flow past a single circular cylinder or sphere. Furthermore, it is more advantageous to employ bi-polar co-ordinates to take into account the treatment of the boundary conditions on the two circular cylinders. This will be especially so for the case of two circular cylinders in close proximity to each other, when the existing analyses in the Oseen approximation lose their accuracies.

Provided that the Reynolds number based on the distance between the axes of the cylinders is sufficiently small, the flow in a region near the obstacles is governed by the Stokes equations in the first approximation. In this paper we consider the description of this Stokes flow in terms of bi-polar co-ordinates, which will assist the derivation of analytical expressions for the forces and moments exerted on the obstacles at small Reynolds numbers, when the method of matched asymptotic expansions is appropriate for our problem. To make this possible, it is, of course, assumed that the method for solving a general Stokes-flow problem around two circular cylinders is established. For three-dimensional Stokes flows around two spheres, a method has been well established since the pioneering works of Stimson & Jeffery (1926) and O'Neill (1964);

but this is not generally the case for Stokes flows around two circular cylinders in two-dimensional problems. There still seems to be some ambiguity in the existing treatment of Stokes flows using bi-polar co-ordinates. The main purpose of this paper is to present a resolution of these difficulties through the analysis of our particular problem.

One fairly general solution of the two-dimensional Stokes equations was found in bi-polar co-ordinates by Jeffery (1922) in his analysis of the Stokes flow between two eccentric circular cylinders rotating about their axes. This flow region is bounded, but applications of the solution to Stokes flow around two circular cylinders in an unbounded fluid region were made by Raasch (1961) and Wakiya (1975) to calculate the forces and moments acting on the obstacles in a simple shearing flow. Wakiya also treated the Stokes flow generated by a certain class of rigid-body motions of two circular cylinders in an otherwise stationary infinite fluid. As pointed out by Jeffery himself, however, his solution is actually applicable only for those flows in which the resultant force on the system of two circular cylinders vanishes identically. Therefore, simple use of Jeffery's solution without modification cannot describe correctly our Stokes flow in which the sum of the drag forces exerted on the obstacles apparently does not vanish. The origin of such an unsatisfactory property of the solution is examined, and necessary modifications are made to obtain appropriate Stokes solutions as the inner expansion in an application of the method of matched asymptotic expansions.

In terms of the modified solution we first calculate the drag, lift and moment coefficients of the cylinders to $O(1)$ in the Reynolds number. They show essentially the same properties as those for the Stokes flow past two spheres, except for their dependences on the Reynolds number due to the singular nature of the two-dimensional problem. In particular they contain no effects of fluid inertia in the flow near the cylinders. However, since the effects are interesting on physical grounds and actually not negligible, we attempt to make an extended analysis to incorporate them partially. It will be found that, to $O(R)$ in the Reynolds number, they can be derived from the Stokes flow around the cylinders as well, but in this case in a purely stretching flow with its second principal axis parallel to the uniform stream. In such a flow the cylinders experience lateral forces which are neglected in the analysis to $O(1)$.

Although, in this paper, the discussion is made exclusive of the derivation of the above hydrodynamic characteristics, it is worth mentioning the interesting patterns of streamlines appearing in the cusp region between the two adjacent circular cylinders. The existence of eddies in Stokes flows was discovered by Dean & Montagnon (1949) for the flow region within a corner formed by two intersecting planes at a sufficiently small angle. This was verified in analytical manner by Moffatt (1964), and he showed the formation of an infinite sequence of eddies whose strength diminishes exponentially approaching the corner. Since then a number of studies of this topic have been performed, examining a variety of other flow geometries and the way in which Moffatt eddies can be formed there. In particular, Davis *et al.* (1976) examined the process of formation of Moffatt eddies for the axisymmetric Stokes flow past two equal spheres as the spacing between them is decreased. They found that the repeating process of the formation of a pair of eddies attached to the facing sides of the spheres and the coalescence of the eddies at the midpoint of the spheres, as the spacing between the spheres decreases, results finally in the Moffatt eddy pattern in the neighbourhood of the point of contact of the spheres. On the other hand, as for our two-dimensional problem of the Stokes flow past two equal circular cylinders, Dorrepaal & O'Neill (1979) have treated a special case of the cylinders with the plane containing their axes perpendicular to the uniform stream, and found as a new feature the formation of free eddies in the flow not attached to any wall. Although there seemingly remains a similar problem for two cylinders in a uniform stream impinging at arbitrary incidence, the flow structure near to close cylinders would generally be a combination of free eddies and Moffatt-type eddies in the region where the cylinders are closest. For further analyses of related problems, and for more complete reviews of the subject, see Jeffrey & Sherwood (1980), Moffatt & Duffy (1980) and Jeffrey & Onishi (1981).

2. Presentation of the problem

The geometry of the flow to be considered is sketched in figure 1. A uniform stream with speed W at infinity impinges on two equal circular cylinders of radius a, and with parallel axes separated by a distance d. We denote by σ the angle between the directions of the uniform stream and of the positive x-axis in the plane containing the axes of the cylinders.

We shall employ the polar co-ordinates (r, θ) as well as the Cartesian co-ordinates (x, y). The co-ordinates $\mathbf{r} \equiv (x, y)$ and r are assumed to be non-dimensionalized by d, and the stream function ψ by Wd. The Reynolds number is accordingly defined by

$$R = Wd/\nu, \tag{1}$$

which is assumed to be much smaller than unity.

Our problem is, in the limiting cases of $R \rightarrow 0$, to find an approximate solution to the Navier–Stokes equation

$$\nabla^4\psi = RJ(\psi, \nabla^2\psi), \tag{2}$$

with the no-slip condition on each cylinder and a uniform stream at infinity. A general approach to this problem has been developed mainly through studies of the low-Reynolds-number flow past a single circular cylinder or sphere. It is the method of matched asymptotic expansions due to Kaplun (1957) and Proudman & Pearson (1957).

In the method, the space round the cylinders is divided into two separate but overlapping regions, and an appropriate perturbation scheme relevant for each region is considered.

(i) In the inner region which satisfies the condition $Rr \leqslant O(1)$, the Stokes equations are the first good approximations to the full Navier–Stokes equations, and the stream function is expanded in the following form:

$$\psi(\mathbf{r}; R) = \sum_{n=1}^{\infty} f_n(R)\psi_n(\mathbf{r}), \tag{3}$$

where

$$f_{n+1}/f_n \rightarrow 0 \quad \text{as} \quad R \rightarrow 0.$$

This inner expansion should be valid asymptotically as $R \rightarrow 0$ for fixed $\mathbf{r}$, and each function ψ_n involved must satisfy the no-slip conditions on both cylinders.

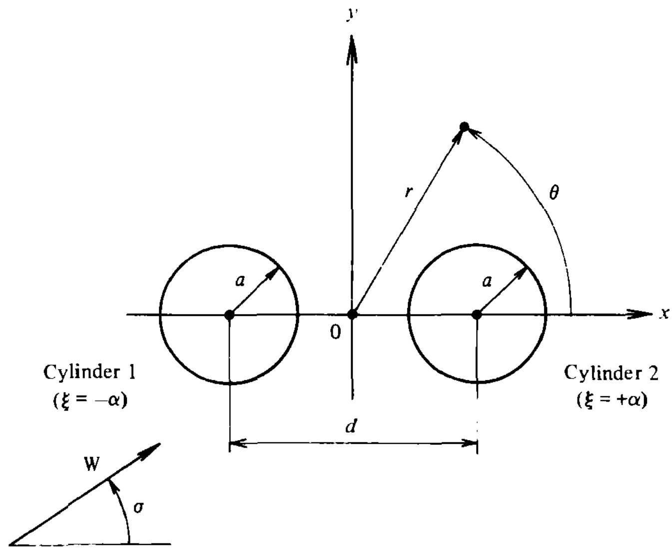

FIGURE 1. Flow geometry and co-ordinate systems.

(ii) In the outer region $Rr \geqslant O(1)$, where inertia forces must be taken into account, the disturbed flow is described in the first approximation by the Oseen equations. In terms of the outer variables defined by

$$\tilde{\mathbf{r}} = R\mathbf{r}, \quad \Psi = R\psi \qquad (4a, b)$$

we here consider the expansion

$$\Psi(\tilde{\mathbf{r}}; R) = \tilde{r}\sin(\theta-\sigma) + \sum_{n=1}^{\infty} F_n(R)\Psi_n(\tilde{\mathbf{r}}). \qquad (5)$$

This outer expansion should be valid asymptotically as $R \to 0$ for fixed $\tilde{\mathbf{r}}$. Since the leading term represents the uniform stream, each function Ψ_n for the disturbed flow must vanish at infinity.

The expansions (3) and (5) are supposed to be derived from the same exact solution of the Navier–Stokes equations, so they must match each other in the region $R \sim O(1)$ where the two regions overlap. This matching condition yields further boundary conditions for each expansion, and enables us to determine its successive terms alternately.

Now, since the flow near the obstacles is dominantly effected by their geometry, it will be advantageous in its description to use bi-polar co-ordinates (ξ, η) defined by

$$x = r\cos\theta = \frac{c}{d}\frac{\sinh\xi}{\cosh\xi - \cos\eta}, \quad y = r\sin\theta = \frac{c}{d}\frac{\sin\eta}{\cosh\xi-\cos\eta} \qquad (6)$$

$$(-\infty < \xi < +\infty, \quad -\pi < \eta \leqslant +\pi)$$

in which the two equal circular cylinders are expressed by the two co-ordinate surfaces $\xi = \pm\alpha \, (\alpha > 0)$ satisfying the conditions

$$a = c\,\mathrm{cosech}\,\alpha, \quad d = 2c\coth\alpha. \qquad (7)$$

On the other hand, the flow far from the obstacles should not depend critically on their geometry. The presence of the obstacles plays first of all a role as a point-force source, and the equivalent flow could be realized by allowing an appropriate force to act on the fluid at the origin. Hence for the description of the far flow field it will be sufficient to consider those solutions with singularities at the origin, and the most suitable co-ordinates are the polar co-ordinates (r, θ).

It is therefore natural to employ the bi-polar and polar co-ordinates in the inner and outer regions respectively, and to use, in applying the matching condition, a transformation between the two co-ordinate systems such that it is asymptotically valid in the overlapping region as $R \to 0$. This transformation can be derived from the asymptotic form of (6) as $r \to \infty$. In fact, since infinity ($r = \infty$) corresponds to the point ($\xi = 0$, $\eta = 0$) and about this point (6) can be approximated by

$$x = r\cos\theta = 2\frac{c}{d}\frac{\xi}{\xi^2+\eta^2}, \quad y = r\sin\theta = 2\frac{c}{d}\frac{\eta}{\xi^2+\eta^2}, \tag{8}$$

then we have, after the substitution of these into (4*a*),

$$\xi = 2R\frac{c}{d}\frac{\cos\theta}{\tilde{r}}, \quad \eta = 2R\frac{c}{d}\frac{\sin\theta}{\tilde{r}}, \tag{9}$$

where $\tilde{r}$ should be regarded as $O(1)$ (and the omitted terms are $O(R^3)$, and are not relevant to our later discussion).

3. Oseen solution

The general solution of the Oseen equations in polar co-ordinates (r, θ) has been presented by Tomotika & Aoi (1950). The relevant part in the solution for our later discussion is the same as that used in the matched-asymptotic analysis of the low-Reynolds-number flow past a single circular cylinder, and is given by

$$\left.\begin{aligned} \Psi^{(h)} &= G\sum_{n=1}^{\infty}\lambda_n(\tfrac{1}{2}\tilde{r})\tilde{r}\,\frac{\sin n(\theta-\sigma)}{n} + H\sum_{n=1}^{\infty}\chi_n(\tfrac{1}{2}\tilde{r})\tilde{r}\,\frac{\cos n(\theta-\sigma)}{n}, \\ \text{with}\quad \lambda_n &= 2K_1I_n + K_0(I_{n-1}+I_{n+1}), \quad \chi_n = K_0(I_{n-1}-I_{n+1}), \end{aligned}\right\} \tag{10}$$

where I_n and K_n are respectively the first and second kinds of modified Bessel functions of order n.

4. Stokes solution

In this section we examine the general form of the Stokes solution that is appropriate to describe the flow in the inner region. It would be better to start with the derivation of the general solution to the two-dimensional Stokes equations in bi-polar co-ordinates. The process will reveal the origin of the difficulties encountered in the application of the existing bi-polar co-ordinate Stokes solution to our problem, and will guide the way to overcome them.

Given the general solution f of the Laplace equation

$$\nabla^2 f = 0 \tag{11}$$

we can construct the general solution ψ of the bi-harmonic equation $\nabla^4\psi = 0$ in the linear combination of the functions f, xf, yf and $r^2 f$. Let us consider this for the general Stokes solution in bi-polar co-ordinates (ξ, η), in which (11) is written in the form

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = 0. \tag{12}$$

We first take up the separable form of the general solution f to (12). The solution, which satisfies the cyclic condition in η, is easily obtained as

$$f = \mathscr{A}\xi + \sum_{n=1}^{\infty} (\mathscr{a}_n \cosh n\xi + \mathscr{b}_n \sinh n\xi) \cos n\eta + \sum_{n=1}^{\infty} (\mathscr{a}'_n \cosh n\xi + \mathscr{b}'_n \sinh n\xi) \sin n\eta, \tag{13}$$

where we have omitted the constant term on account of the arbitrariness of the additive constant in the stream function. The Stokes solution then becomes

$$\begin{aligned}\phi = \phi_J = {} & A\xi(\cosh\xi - \cos\eta) + B\sinh\xi + C\xi\sinh\xi + D\xi\sin\eta \\ & + \sum_{n=1}^{\infty} [a_n \cosh(n+1)\xi + b_n \sinh(n+1)\xi + c_n\cosh(n-1)\xi + d_n \sinh(n-1)\xi]\cos n\eta \\ & + \sum_{n=1}^{\infty} [a'_n \cosh(n+1)\xi + b'_n \sinh(n+1)\xi + c'_n\cosh(n-1)\xi + d'_n \sinh(n-1)\xi]\sin n\eta,\end{aligned} \tag{14}$$

when we put

$$\psi = \frac{c}{d}\frac{\phi}{\cosh\xi - \cos\eta}. \tag{15}$$

The solution (14) was first presented by Jeffery (1922) to describe the Stokes flow between two eccentric circular cylinders (expressible by two co-ordinate surfaces $\xi = \xi_1$ and ξ_2 (>0)) rotating about their axes.

The solution (14), which was complete for the flow in the bounded region above, is, however, not complete over the whole fluid region exterior to our circular cylinders. Its completeness will break down at infinity. To see this, let us examine the asymptotic form of the stream function (15) with (14) as $r \to \infty$. By use of (8) we have

$$\psi = \tfrac{1}{2}r^2 \frac{d}{c}\sum_{n=1}^{\infty}(a_n + c_n) + x\left[B + \sum_{n=1}^{\infty}\{(n+1)b_n + (n-1)d_n\}\right] + y\sum_{n=1}^{\infty} n(a'_n + c'_n), \tag{16}$$

where we have omitted those terms that converge at infinity. The first term expresses a rigid-body rotation, and the second and third, uniform translations in the y- and x-directions respectively. No other diverging terms appear, implying that (14) cannot describe any flow at infinity but only rigid-body motions. Besides, the rigid-body motions expressed by (16) cannot be arbitrary. This is evident from the very fact that (14) was complete in Jeffery's analysis cited above. The arbitrary constants appearing in the solution (14), and therefore the coefficients in (16), are all determined irrespectively of the boundary condition at infinity if the boundary conditions on 'any' two circular cylinders are specified.

It will be also of interest to examine the forces acting on the cylinders. Generally, the x- and y-components of the forces on the two equal circular cylinders $\xi = \pm\alpha$ and the moments about their axes in the anticlockwise direction, in the Stokes approximation, are calculated in terms of the function ϕ in (15) as follows:

$$\left.\begin{aligned} F_x^{(\pm)} &= \pm\mu W\int_0^{2\pi} d\eta \sin\eta\left[\frac{\partial^3\phi}{\partial\xi^3} + 4\frac{\partial\phi}{\partial\xi}\right]_{\xi=\pm\alpha}, \\ F_y^{(\pm)} &= \pm\mu W\int_0^{2\pi} d\eta\left[\left(\frac{\partial^3\phi}{\partial\xi^3} - \frac{\partial\phi}{\partial\xi}\right)\sinh\xi - \left(\frac{\partial^2\phi}{\partial\xi^2} - \phi\right)\cosh\xi\right]_{\xi=\pm\alpha}, \\ M^{(\pm)} &= \mp\mu W a\int_0^{2\pi} d\eta\left[\frac{\partial^2\phi}{\partial\xi^2} - \phi\right]_{\xi=\pm\alpha}, \end{aligned}\right\} \tag{17}$$

where $\mu = \rho\nu$ denotes the viscosity of fluid, and the multiplying factor W is due to the non-dimensionalization of the stream function by Wd. The substitution of (14) into (17) leads to the expressions

$$F_x^{(\pm)} = \pm 4\pi\mu DW, \quad F_y^{(\pm)} = \mp 4\pi\mu CW, \tag{18a, b}$$

$$M^{(\pm)} = -4\pi\mu a[\pm A \sinh\alpha + C\cosh\alpha]W, \tag{18c}$$

and from (18a, b) the resultant force on the system of two cylinders is found to vanish identically. This implies that simple use of (14) does not allow us to analyse correctly those flows with a non-zero resultant force on the system. In particular, this is the case for our Stokes flow.

Such unsatisfactory properties of the solution (14) come apparently from the choice of the separable form of the solution (13) to the basic Laplace equation (12). In this respect we should note that the general polar-co-ordinate Stokes solution

$$\begin{aligned}\psi = {} & 2Lr^2 \ln r + 2(P\cos\theta - Q\sin\theta)\, r\ln r + 2\Gamma \ln r \\ & + \sum_{n=1}^{\infty} [(a_n r^{n+2} + b_n r^n)\cos n\theta + (c_n r^{n+1} + d_n r^n)\sin n\theta],\end{aligned} \tag{19}$$

which is derived using the same procedure, is complete over the whole fluid region interior or exterior to a single circular cylinder.

The diverging terms missing from the harmonic function (13), which are not in the form of separation of variables, can be obtained by differentiating with respect to the variables ξ and η the function $\ln(\cosh\xi - \cos\eta)$ in the fundamental solution of the Laplace equation

$$\ln \frac{d}{c} r = \tfrac{1}{2}\ln(\cosh\xi + \cos\eta) - \tfrac{1}{2}\ln(\cosh\xi - \cos\eta). \tag{20}$$

Among the diverging Stokes solutions constructed with their aid, the most important are the terms

$$[L(\cosh\xi + \cos\eta) + P\sinh\xi - Q\sin\eta + \Gamma(\cosh\xi - \cos\eta)]\ln(\cosh\xi - \cos\eta), \tag{21}$$

which correspond respectively to the first four terms of (19) in their behaviours at infinity. They contribute to the hydrodynamic characteristics of the two cylinders. In fact the expressions for the forces and moments acting on the cylinders now become

$$\left.\begin{aligned} F_x^{(\pm)} &= 4\pi\mu(-Q \pm D)W, \quad F_y^{(\pm)} = 4\pi\mu(-P \mp C)W, \\ M^{(\pm)} &= -4\pi\mu a[(L + \Gamma \pm A)\sinh\alpha + (C \pm P)\cosh\alpha]\, W, \end{aligned}\right\} \tag{22}$$

and it is found that (14) supplemented with (19) enables us to treat the case of non-zero resultant force on both cylinders.

The terms involving the coefficients P and Q are called the Stokeslets, and they are related with the y- and x-components of the resultant force on the two cylinders. Now we consider the form of the solution that describes our Stokes flow past two equal circular cylinders. At this stage it will be instructive to recall a fundamental property of the Stokes flow past a single circular cylinder; the force exerted on the cylinder appears in connection with the Stokeslet $2(Px - Qy)\ln r$ that causes the logarithmic divergence of the velocity at infinity. Such a diverging behaviour in the far Stokes flow is, however, not restricted to this single-circular-cylinder case, but holds for any finite system of obstacles in a uniform stream. This can be seen by considering the surface integral of the forces on a large circular cylinder that includes the system in it. Thus we assume

$$\begin{aligned}\phi = {} & [P\sinh\xi - Q\sin\eta + \Gamma(\cosh\xi - \cos\eta)]\ln(\cosh\xi - \cos\eta) \\ & + A\xi(\cosh\xi - \cos\eta) + B\sinh\xi + C\xi\sinh\xi + D\xi\sin\eta \\ & + \sum_{n=1}^{\infty}[a_n\cosh(n+1)\xi + b_n\sinh(n+1)\xi + c_n\cosh(n-1)\xi + d_n\sinh(n-1)\xi]\cos n\eta \\ & + \sum_{n=1}^{\infty}[a'_n\cosh(n+1)\xi + b'_n\sinh(n+1)\xi + c'_n\cosh(n-1)\xi \\ & + d'_n\sinh(n-1)\xi]\sin n\eta. \end{aligned} \tag{23}$$

The boundary conditions to be satisfied on the two stationary cylinders $\xi = \pm\alpha$ are

$$\frac{\partial\psi}{\partial\eta}(\pm\alpha) = 0, \quad \frac{\partial\psi}{\partial\xi}(\pm\alpha) = 0. \tag{24a, b}$$

The first condition (24a) implies that the stream function takes a constant value $\psi(\pm\alpha) = (c/d)K_{\pm}$, say, on each cylinder $\xi = \pm\alpha$. Hence the conditions can be expressed in terms of the function ϕ as

$$\phi(\pm\alpha) = K_{\pm}(\cosh\xi - \cos\eta), \quad \frac{\partial\phi}{\partial\xi}(\pm\alpha) = \pm K_{\pm}\sinh\alpha. \tag{25}$$

Substituting (23) into (25), and equating the Fourier components on each side after use is made of the expansion

$$\ln(\cosh\alpha - \cos\eta) = \alpha - \ln 2 - 2\sum_{n=1}^{\infty}\frac{e^{-n\alpha}}{n}\cos n\eta, \tag{26}$$

we obtain the following sets of simultaneous equations for the arbitrary constants in (23) as well as the $K_{\pm}$ introduced above:

$$\left.\begin{aligned} A\alpha\cosh\alpha + B\sinh\alpha - K\cosh\alpha &= -P(\alpha - \ln 2)\sinh\alpha, \\ A(\alpha\sinh\alpha + \cosh\alpha) + B\cosh\alpha - K\sinh\alpha &= -P[(\alpha - \ln 2)\cosh\alpha + \sinh\alpha], \\ -A\alpha + b_1\sinh 2\alpha + K &= P(1 - e^{-2\alpha}), \\ -A + 2b_1\cosh 2\alpha &= 2Pe^{-2\alpha}, \end{aligned}\right\} \tag{27a}$$

$$\left.\begin{aligned} b_n\sinh(n+1)\alpha + d_n\sinh(n-1)\alpha &= P\frac{1}{n}[e^{-(n-1)\alpha} - e^{-(n+1)\alpha}], \\ (n+1)b_n\cosh(n+1)\alpha + (n-1)\cosh(n-1)\alpha & \\ &= P\frac{1}{n}[(n+1)e^{-(n+1)\alpha} - (n-1)e^{-(n-1)\alpha}], \end{aligned}\right\} \quad (n \geqslant 2) \tag{27b}$$

$$\left.\begin{aligned} a'_1\cosh 2\alpha + c'_1 &= -Q(\alpha - \ln 2 + \tfrac{1}{2}e^{-2\alpha}), \\ 2a'_1\sinh 2\alpha &= -Q(1 - e^{-2\alpha}), \end{aligned}\right\} \tag{27c}$$

$$\left.\begin{aligned} a'_n\cosh(n+1)\alpha + c'_n\cosh(n-1)\alpha &= Q\left[\frac{e^{-(n-1)\alpha}}{n-1} - \frac{e^{-(n+1)\alpha}}{n+1}\right], \\ (n+1)a'_n\sinh(n+1)\alpha + (n-1)c'_n\sinh(n-1)\alpha &= Q[e^{-(n+1)\alpha} - e^{-(n-1)\alpha}]. \end{aligned}\right\} \quad (n \geqslant 2) \tag{27d}$$

Here $K_+ = K_- \equiv K$, and the arbitrary constants other than those appearing in the above equations are all equal to zero. Then (23) is found to be expressible in the form

$$\phi = P[\sinh\xi \ln(\cosh\xi - \cos\eta) + \hat{A}\xi(\cosh\xi - \cos\eta) + \hat{B}\sinh\xi$$
$$+ \sum_{n=1}^{\infty} \{\hat{b}_n \sinh(n+1)\xi + d_n \sinh(n-1)\xi\}\cos n\eta]$$
$$-Q[\sin\eta \ln(\cosh\xi - \cos\eta) + \sum_{n=1}^{\infty} \{\hat{a}'_n \cosh(n+1)\xi + \hat{c}'_n \cosh(n-1)\xi\}\sin n\eta], \tag{28}$$

with, from the solutions to (27),

$$\hat{A} = \frac{A}{P} = -\frac{2+\cosh 2\alpha}{\sinh 2\alpha}, \qquad \hat{b}_n = \frac{b_n}{P} = \frac{1}{n}\frac{(n-1) - ne^{-2\alpha} + e^{-2n\alpha}}{n\sinh 2\alpha - \sinh 2n\alpha}, \tag{29a, b}$$

$$\hat{B} = \frac{B}{P} = \ln 2 - \alpha + \tfrac{3}{2}\coth\alpha, \qquad \hat{d}_n = \frac{d_n}{P} = \frac{1}{n}\frac{(n+1) - ne^{2\alpha} - e^{-2n\alpha}}{n\sinh 2\alpha - \sinh 2n\alpha}, \tag{29c, d}$$

$$\hat{b}_1 = \frac{b_1}{P} = -\frac{1+2e^{-2\alpha}}{2\sinh 2\alpha}, \qquad \hat{K} = \frac{K}{P} = -\frac{2+\cosh 2\alpha}{\sinh 2\alpha} + \frac{3}{2}, \tag{29e, f}$$

$$\hat{a}'_1 = \frac{a'_1}{-Q} = \frac{-1}{1+e^{2\alpha}}, \qquad \hat{a}'_n = \frac{a'_n}{-Q} = \frac{1}{n+1}\frac{-(n+1) + ne^{-2\alpha} + e^{-2n\alpha}}{n\sinh 2\alpha + \sinh 2n\alpha}, \tag{29g, h}$$

$$\hat{c}'_1 = \frac{c'_1}{-Q} = \ln 2 - \alpha - \tfrac{1}{2}e^{-2\alpha} + \frac{\cosh 2\alpha}{1+e^{2\alpha}}, \qquad \hat{c}'_n = \frac{c'_n}{-Q} = \frac{1}{n-1}\frac{-(n-1) + ne^{2\alpha} - e^{-2n\alpha}}{n\sinh 2\alpha + \sinh 2n\alpha} \tag{29i, j}$$

Note that the coefficients in (29) are functions only of the parameter α. They are therefore determined from (7) in terms of the non-dimensional distance between the axes of the two equal circular cylinders, $\zeta = d/2a$. On the other hand, to determine the unknown multiplying factors P and Q, it is necessary to apply the matching condition between the Stokes solution (28) and the Oseen solution (10) in §5.

5. Matching to $O(1)$

As mentioned earlier, the dominant feature of the far Stokes flow past the cylinders is the logarithmic divergence of the velocity due to the Stokeslet. This fact makes it possible to perform the matching between the inner and outer expansions to $O(1)$ in a similar way to the case for a single circular cylinder (see Proudman & Pearson 1957; Van Dyke 1975). In the expansions (3) and (5) we can put

$$f_n = F_n = (\ln R)^{-n}, \tag{30}$$

and take as ψ_n and $\Psi_n^{(h)}$ (i.e. the complementary solution to the perturbation equation for $\Psi_n = \Psi_n^{(h)} + \Psi_n^{(p)}$) the Stokes and Oseen solutions (28) and (10) in which the multiplying factors P, Q, G and H are replaced by the same symbols with subscript n. Then, the problem is to determine P_n and Q_n by applying the matching condition, which demands that the asymptotic form of the inner expansion (3) as $r \to \infty$ should match that of the outer expansion (5) as $\tilde{r} \to 0$ in the limit $R \to 0$.

The particular solution $\Psi_n^{(p)}$ of the perturbation equation for Ψ_n appears for $n \geqslant 2$, but those that have contributions to the inner expansion of order-unity are found to be for $n \geqslant 3$. The calculation, however, becomes very complex, and is practically impossible to do for all values of n. In the following treatment of the matching condition, we neglect the contributions from $\Psi_n^{(p)}$, and consider the matching between ψ_n and $\Psi_n^{(h)}$ to obtain, in §6, the explicit expressions for the forces and moments acting on the cylinders, which would be equivalent to those derived from the Oseen equations to $O(1)$. This may be significant in considering the fact that there is no separable form of the Oseen solution in bi-polar co-ordinates.

About infinity the function $\psi_n(\xi, \eta)$ is written in terms of the outer variables (9) as

$$R\psi_n = 2P_n[\tilde{r}\ln\tilde{r} - (\ln R + p)\tilde{r}]\cos\theta - 2Q_n[\tilde{r}\ln\tilde{r} - (\ln R + q)\tilde{r}]\sin\theta, \tag{31}$$

where we have omitted the terms of order less than unity, and put

$$p = \ln\frac{c}{d} + \frac{1}{2}\left[\ln 2 + \hat{B} + \sum_{n=1}^{\infty}\{(n+1)\hat{b}_n + (n-1)d_n\}\right], \tag{32a}$$

$$q = \ln\frac{c}{d} + \frac{1}{2}\left[\ln 2 + \sum_{n=1}^{\infty} n(\hat{a}_n' + \hat{c}_n')\right]. \tag{32b}$$

About the origin the function $\Psi_n^{(h)}(\tilde{r}, \theta)$, on the other hand, takes the form

$$\begin{aligned}\Psi_n^{(h)} = {} & [(G_n\sin\sigma - H_n\cos\sigma)\tilde{r}\ln\tilde{r} - \{G_n(g+\tfrac{1}{2})\sin\sigma - H_n(g-\tfrac{1}{2})\cos\sigma\}\tilde{r}]\cos\theta \\ & - [(G_n\cos\sigma + H_n\sin\sigma)\tilde{r}\ln\tilde{r} - \{G_n(g+\tfrac{1}{2})\cos\sigma + H_n(g-\tfrac{1}{2})\sin\sigma\}r]\sin\theta,\end{aligned} \tag{33}$$

where

$$g = \tfrac{1}{2} - \gamma = 2\ln 2, \tag{34}$$

and γ is Euler's constant.

The expansions (3) and (5), substituted by (31) and (33) respectively, are to be matched to each other in the limit $R \to 0$, so that the following set of simultaneous equations in recursive forms must hold for the unknown constants P_n, Q_n, G_n and H_n $(n \geqslant 1)$:

$$\left.\begin{aligned} G_n\sin\sigma - H_n\cos\sigma &= 2P_n, \\ G_n\cos\sigma + H_n\sin\sigma &= 2Q_n, \\ G_n(g+\tfrac{1}{2})\sin\sigma - H_n(g-\tfrac{1}{2})\cos\sigma &= 2P_{n+1} + 2pP_n, \\ G_n(g+\tfrac{1}{2})\cos\sigma + H_n(g-\tfrac{1}{2})\sin\sigma &= 2Q_{n+1} + 2qQ_n.\end{aligned}\right\} \tag{35}$$

The solutions are

$$\begin{pmatrix}P_n\\Q_n\end{pmatrix} = \frac{t_+^{n-1} - t_-^{n-1}}{t_+ - t_-}\begin{pmatrix}P_2\\Q_2\end{pmatrix} + \frac{t_+t_-^{n-1} - t_-t_+^{n-1}}{t_+ - t_-}\begin{pmatrix}P_1\\Q_1\end{pmatrix}, \tag{36}$$

with

$$t_\pm = g - \tfrac{1}{2}(p+q) \pm \tfrac{1}{2}[(p-q)^2 + 2(p-q)\cos 2\sigma + 1]^{\frac{1}{2}}.$$

And we also have the relations

$$\left.\begin{aligned} P_2 &= (g - p - \tfrac{1}{2}\cos 2\sigma)P_1 + \tfrac{1}{2}Q_1\sin 2\sigma, \\ Q_2 &= (g - q + \tfrac{1}{2}\cos 2\sigma)Q_1 + \tfrac{1}{2}P_1\sin 2\sigma.\end{aligned}\right\} \tag{37}$$

The initial values P_1 and Q_1 for (36) with (37) are determined from the matching condition of the leading term of the inner expansion with the uniform streaming term of the outer expansion, yielding

$$2P_1 = \sin\sigma, \quad 2Q_1 = \cos\sigma. \tag{38}$$

This completes the determination of all the unknown multiplying factors P_n and Q_n under consideration, and the inner expansion thus determined can be expressed into a compact form by taking the summation over all n. It becomes the Stokes solution (28), with the multiplying factors P and Q given by

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \sum_{n=1}^{\infty} \frac{1}{(\ln R)^n} \begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \frac{\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} + (\ln R - t_+ - t_-) \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}}{(\ln R - t_+)(\ln R - t_-)}$$

$$= \frac{\frac{1}{2}\begin{pmatrix} (\ln R + \frac{1}{2} - g + q)\sin\sigma \\ (\ln R + \frac{1}{2} - g + p)\cos\sigma \end{pmatrix}}{[\ln R - g + \frac{1}{2}(p+q)]^2 - \frac{1}{4}[(p-q)^2 + 2(p-q)\cos 2\sigma + 1]}, \tag{39}$$

6. Interaction characteristics to $O(1)$

The expressions for the drag, lift and moment coefficients of each cylinder are obtained to order unity by substituting (39) into (22) as follows:

$$C_{\mathrm{D}}^{(\pm)} = \frac{F_x^{(\pm)}\cos\sigma + F_y^{(\pm)}\sin\sigma}{\frac{1}{2}\rho W^2(2a)}$$

$$= -\frac{8\pi}{R^*} \frac{2\ln R^* + 1 - 2g + p^* + q^* + (p^* - q^*)\cos 2\sigma}{[2\ln R^* - 2g + p^* + q^*]^2 - [(p^*-q^*)^2 + 2(p^*-q^*)\cos 2\sigma + 1]}, \tag{40a}$$

$$C_{\mathrm{L}}^{(\pm)} = \frac{F_x^{(\pm)}\sin\sigma - F_y^{(\pm)}\cos\sigma}{\frac{1}{2}\rho W^2(2a)}$$

$$= -\frac{8\pi}{R^*} \frac{(p^* - q^*)\sin 2\sigma}{[2\ln R^* - 2g + p^* + q^*]^2 - [(p^*-q^*)^2 + 2(p^*-q^*)\cos 2\sigma + 1]}, \tag{40b}$$

$$C_{\mathrm{M}}^{(\pm)} = \frac{M^{(\pm)}}{\frac{1}{2}\rho W^2(2a)^2}$$

$$= \mp\frac{4\pi}{R^*} h \frac{(2\ln R^* + 1 - 2g + q^*)\sin\sigma}{[2\ln R^* - 2g + p^* + q^*]^2 - [(p^*-q^*)^2 + 2(p^*-q^*)\cos 2\sigma + 1]}, \tag{40c}$$

with

$$h = A\sinh\alpha + \cosh\alpha. \tag{41}$$

Here we have introduced the Reynolds number

$$R^* = W(2a)/\nu, \tag{42}$$

and put

$$p^* = p + \ln\zeta, \quad q^* = q + \ln\zeta, \tag{43a, b}$$

in accordance with the change of the reference Reynolds number from R to R^*. The expression (40a) should especially be compared with the corresponding drag coefficient for an isolated circular cylinder:

$$C_{\mathrm{D}}^* = -\frac{8\pi}{R^*}\frac{1}{\ln\frac{1}{2}R^* - g}, \tag{44}$$

The effects of the interaction between the two equal circular cylinders are found to be characterized by the three non-dimensional quantities p^*, q^* and h, which are functions only of the parameter ζ as shown in figure 2.

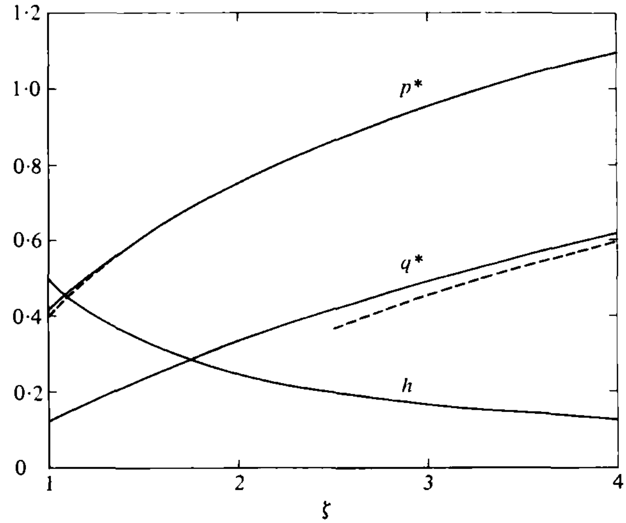

FIGURE 2. The non-dimensional quantities p^*, q^* and h as functions of the parameter $\zeta = d/2a$. The dashed curves are the asymptotes (45) for $\zeta \to \infty$.

Let us here consider the following three limiting cases.

(i) $\zeta \to \infty$. For the far-apart cylinders we obtain the asymptotes

$$\left.\begin{aligned} p^* &= \tfrac{1}{2}\ln\zeta - \tfrac{1}{2}\ln 2 + \tfrac{3}{4}, \\ q^* &= \tfrac{1}{2}\ln\zeta - \tfrac{1}{2}\ln 2 + \tfrac{1}{4}. \end{aligned}\right\} \tag{45}$$

Hence we have the expressions

$$\left.\begin{aligned} C_D &\simeq -\frac{8\pi}{R^*}\frac{1}{2\ln\frac{1}{2}R^* - 2g + \frac{1}{2} + \ln\zeta} \quad (\sigma = 0), \\ C_D &\simeq -\frac{8\pi}{R^*}\frac{1}{2\ln\frac{1}{2}R^* - 2g - \frac{1}{2} + \ln\zeta} \quad (\sigma = \tfrac{1}{2}\pi). \end{aligned}\right\} \tag{46}$$

These are exactly in agreement with the results of Kuwabara (1957), who solved the Oseen equation by a perturbation method relevant for large ζ under the same condition $R \ll 1$.

(ii) $\zeta \to 1$ ($\alpha \to 0$). It is of interest to examine the values of the quantities p^* and q^* for the cylinders in contact, for which the bi-polar co-ordinates employed become singular, and the convergence of the infinite series in (32) becomes very poor. More appropriate co-ordinates are tangential ones. However, since they can be obtained from the bi-polar co-ordinates by taking the limit $\alpha \to 0$ in (6), we could derive the limiting values of p^* and q^* from (43) directly. This procedure necessitates some manipulation, which we shall describe below.

After the substitution of (29) and (32a), the quantity p^* of (43a) is written as

$$p^* = \ln(\tanh\alpha) - \tfrac{1}{2}\alpha + \tfrac{3}{4}\coth\alpha - \frac{1+2e^{-2\alpha}}{2\sinh 2\alpha} + \sum_{n=2}^{\infty}\frac{1}{n}\frac{n\sinh 2\alpha - (1-e^{-2n\alpha}) - 2n^2\sinh^2\alpha}{n\sinh 2\alpha - \sinh 2n\alpha}, \tag{47}$$

where the expansion

$$\ln(\tanh\alpha) = -2\sum_{n=1}^{\infty}\frac{e^{-2n\alpha}}{2n} \quad (\alpha > 0) \tag{48}$$

is available for the first term on the right-hand side. In the limit $\alpha \to 0$ the contributions from the infinite series can be divided into two parts: one for finite n such that $n\alpha \to 0$, and the other for infinitely large n such that $n\alpha \to s$, a value ranging from 0 to ∞, and the latter is transformed into an integral form. Thus we have

$$p^* = \tfrac{1}{4} - \ln 2 + \int_0^\infty ds \left[\frac{2s(1-s)-(1-e^{-2s})}{s(2s-\sinh 2s)} - \frac{e^{-s}}{s}\right] = 0{\cdot}410. \tag{49}$$

Similarly we obtain

$$q^* = -1 - \ln 2 + \int_0^\infty ds \left[\frac{2s(1+s)+(1-e^{-2s})}{2(2s+\sinh 2s)} - \frac{e^{-s}}{s}\right] = 0{\cdot}112. \tag{50}$$

(iii) $R^* \to 0$. From (40a) and (41) we obtain the relation

$$\lim_{R^*\to 0} \frac{C_{\mathrm{D}}^{(\pm)}}{C_{\mathrm{D}}^*} = \frac{1}{2}. \tag{51}$$

This was first found by Fujikawa (1956) on the basis of the Oseen equations.

It is apparent from the construction of our solution that the asymptote of the drag coefficient (40a) as $R^* \to 0$ is reduced to that derived from the leading term of the inner expansion (3). In this respect it is worthwhile to note the following. Consider generally the low-Reynolds-number flow past any finite system of obstacles. Defining the Reynolds number R appropriately, the solution in the matched-asymptotic-expansion method always gives $f_1 = 1/\ln R$, so that the leading term of the inner expansion might match the uniform stream term of the outer expansion. This implies that the resultant force on the system to $O(1/\ln R)$ always takes the same value. It is equal to the drag force acting on a single circular cylinder in the uniform stream, and when the system is composed of multiple obstacles it must be shared with each component in part. In particular, for our system of two equal circular cylinders, each cylinder must share just half the resultant force with each other by symmetry. This gives an interpretation of (51).

7. Inertial effects

The hydrodynamic characteristics (40) contain no effects of fluid inertia except their dependencies on the Reynolds number due to the singular nature of the two-dimensional problem at hand. They essentially show the same properties as those for the corresponding Stokes flow past two spheres; each cylinder has the same force and there appear no lateral forces acting on the cylinders. Actually, the cylinders for $\sigma = \frac{1}{2}\pi$, for example, will experience not only drag forces but lateral force components mutually repulsive to each other, and a cylinder in the wake of the other will have a smaller drag force. In this section we extend the analysis to incorporate such fluid-inertia effects partially by considering contributions from higher-order terms, which have been neglected in the previous sections.

Strictly speaking, all the terms of order unity must be exhausted before considering the terms of order R. The calculations are, as mentioned earlier, impossible to do in practice. For, since R is transcendentally small compared with any power of $1/\ln R$, we encounter the difficulty of solving an infinite number of perturbation equations with successively complicated inhomogeneous terms. But when exclusive attention is paid to the first term of order R in the inner expansion, the term will be easily determined from the matching condition with the outer expansion available.

Following Proudman & Pearson, let us write the inner and outer expansions in the forms

$$\psi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{R^m}{(\ln R)^n} \psi_{m,n}, \tag{52}$$

$$\Psi = \tilde{r} \sin(\theta - \sigma) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{R^m}{(\ln R)^n} \Psi_{m,n}. \tag{53}$$

Then the terms treated in § 5 are those for $m = 0$, and it is apparent that $\psi_{0,0} = \Psi_{0,0} = 0$.

Now, the re-examination of the matching process in (52) and (53) proves that the contribution to the term $R\psi_{1,0}$ comes from the matching condition with the relevant parts of the terms $(\ln R)^{-1}\Psi_{0,1}$ and $(\ln R)^{-2}\Psi_{0,2}$. In fact, we have

$$\frac{1}{R \ln R}\Psi_{0,1} = -r\sin(\theta-\sigma) + \frac{1}{\ln R}[r \ln r - (g+\tfrac{1}{2})r]\sin(\theta-\sigma)$$
$$- R\tfrac{1}{8}r^2 \sin 2(\theta-\sigma) - \frac{R}{\ln R}\tfrac{1}{8}(r^2 \ln r - gr^2)\sin 2(\theta-\sigma) + O(R^2). \tag{54}$$

Here we should pay attention to the third term of order R. The other contributing term comes from the particular solution of the perturbation equation for $\Psi_{0,2}$, and it is given by $-\frac{1}{16}Rr^2 \sin 2(\theta-\sigma)$. Therefore, in the term $R\psi_{1,0}$ that matches these terms, the stream function $\psi_{1,0}$ must satisfy the condition

$$\psi_{1,0} \to -\tfrac{3}{16}r^2 \sin 2(\theta-\sigma) \quad \text{as} \quad r \to \infty, \tag{55}$$

as well as the no-slip condition on the cylinders. Moreover, the function is found to be governed by the Stokes equations. Hence the stream function $\psi_{1,0}$ represents the Stokes flow around the two cylinders placed in the purely stretching flow (55) with the second principal axis parallel to the uniform stream, and it can be solved in terms of the solution (23) as well.

The ϕ-function version of (55) by (15) becomes

$$\Phi = E_c \frac{\sinh\xi \sin\eta}{\cosh\xi - \cos\eta} + E_s \frac{\sinh^2\xi - \sin^2\eta}{\cosh\xi - \cos\eta}, \tag{56a}$$

with

$$E_c = -\frac{3c}{16d}\cos 2\sigma \quad E_s = \frac{3c}{32d}\sin 2\sigma. \tag{56b, c}$$

Let us assume the flow to be expressible as the superposition of Φ and ϕ in (15). Then, applying the no-slip condition we obtain the function ϕ in the following form:

$$\phi = E_c\left[\hat{D}\xi \sin\eta + \sum_{n=1}^{\infty}\{\hat{b}'_n \sinh(n+1)\xi + d'_n \sinh(n-1)\xi\}\sin n\eta\right]$$
$$+ E_s\left[\hat{\Gamma}(\cosh\xi - \cos\eta)\ln(\cosh\xi - \cos\eta)\right.$$
$$\left. + \hat{C}\xi \sinh\xi + \sum_{n=1}^{\infty}\{\hat{a}_n \cosh(n+1)\xi + \hat{c}_n \cosh(n-1)\xi\}\cos n\eta\right], \tag{57}$$

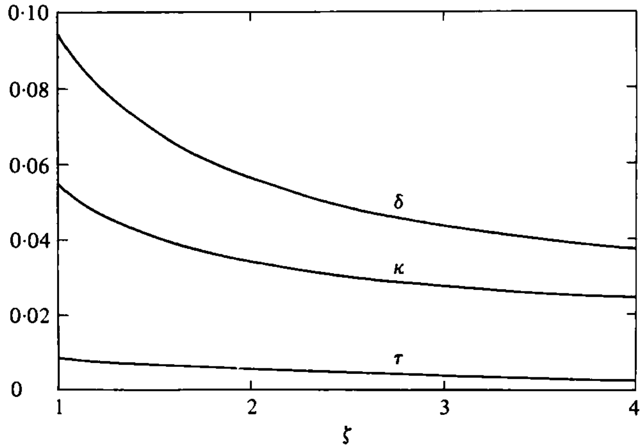

FIGURE 3. The non-dimensional quantities δ, κ and τ as functions of the parameter $\zeta = d/2a$.

where

$$\hat{D} = \frac{D}{E_c} = \frac{1-\cosh 2\alpha}{2\alpha\cosh 2\alpha - \sinh 2\alpha}, \qquad \hat{b}_1' = \frac{b_1'}{E_c} = \frac{1-(1+2\alpha)\,e^{-2\alpha}}{2\alpha\cosh 2\alpha - \sinh 2\alpha}, \tag{58a, b}$$

$$\hat{b}_n' = \frac{b_n'}{E_c} = \frac{-(n-1)+ne^{-2\alpha}-e^{-2n\alpha}}{n\sinh 2\alpha - \sinh 2n\alpha}, \quad \hat{d}_n' = \frac{d_n'}{E_c} = \frac{-(n+1)+ne^{2\alpha}+e^{-2n\alpha}}{n\sinh 2\alpha - \sinh 2n\alpha} \quad (n \geqslant 2), \tag{58c, d}$$

$$\hat{C} = \frac{C}{E_s} = \frac{2}{2\alpha+\sinh 2\alpha} - \frac{2\sinh^2\alpha}{2\alpha+\sinh 2\alpha}\hat{\Gamma}, \tag{58e}$$

$$\hat{a}_1 = \frac{a_1}{E_s} = \frac{2}{e^{4\alpha}-1} + \frac{1}{e^{2\alpha}+1}\hat{\Gamma}, \tag{58f}$$

$$\hat{c}_1 = \frac{c_1}{E_s} = -\frac{4\alpha+4\sinh 2\alpha - \sinh 4\alpha}{2(2\alpha+\sinh 2\alpha)\sinh 2\alpha} + \frac{(2\alpha\cosh 2\alpha+\sinh 2\alpha)\sinh\alpha}{2(2\alpha+\sinh 2\alpha)\cosh\alpha}\hat{\Gamma}, \tag{58g}$$

$$\left.\begin{aligned} \hat{a}_n &= \frac{a_n}{E_s} = \frac{-(n-1)+ne^{-2\alpha}+e^{-2n\alpha}}{n\sinh 2\alpha+\sinh 2n\alpha} + \frac{1}{n(n+1)}\,\frac{(n+1)-ne^{-2\alpha}-e^{-2n\alpha}}{n\sinh 2\alpha+\sinh 2n\alpha}\hat{\Gamma}, && (58h)\\ \hat{c}_n &= \frac{c_n}{E_s} = \frac{-(n+1)+ne^{2\alpha}-e^{-2n\alpha}}{n\sinh 2\alpha+\sinh 2n\alpha} + \frac{1}{n(n-1)}\,\frac{(n-1)-ne^{2\alpha}+e^{-2n\alpha}}{n\sinh 2\alpha-\sinh 2n\alpha}\hat{\Gamma} && (58i)\end{aligned}\right\} \quad (n \geqslant 2).$$

The coefficient $\hat{\Gamma}$, which remains unknown, is determined from the boundary condition (55) at infinity. The rigid-body rotation which appears in the asymptotic form of (57) as $r \to \infty$ (see (16)) must vanish, so that we have the condition

$$\sum_{n=1}^{\infty} (\hat{a}_n + \hat{c}_n) = 0. \tag{59}$$

This gives an equation for $\hat{\Gamma}$, thus determining all the coefficients in (58) completely.

The corrections to the hydrodynamic characteristics (40) to the lowest order in R^* are therefore obtained from (22), and yield

$$C_D^{(\pm)} = \pm\, 8\pi(D\cos\sigma + C\sin\sigma)\,\zeta, \tag{60a}$$

$$C_L^{(\pm)} = \pm\, 8\pi(C\cos\sigma - D\sin\sigma)\,\zeta, \tag{60b}$$

$$C_M^{(\pm)} = -\,4\pi(\Gamma\sinh\alpha + C\cosh\alpha)\,\zeta. \tag{60c}$$

As can be seen from (56*b*), (58) and (59), the quantities $\delta \equiv -D/\cos 2\sigma$, $\kappa \equiv C/\sin 2\sigma$ and $\tau \equiv (\Gamma\sinh\alpha + C\cosh\alpha)/\sin 2\sigma$ are functions only of the parameter α, and their dependences on the parameter ζ are plotted in figure 3.

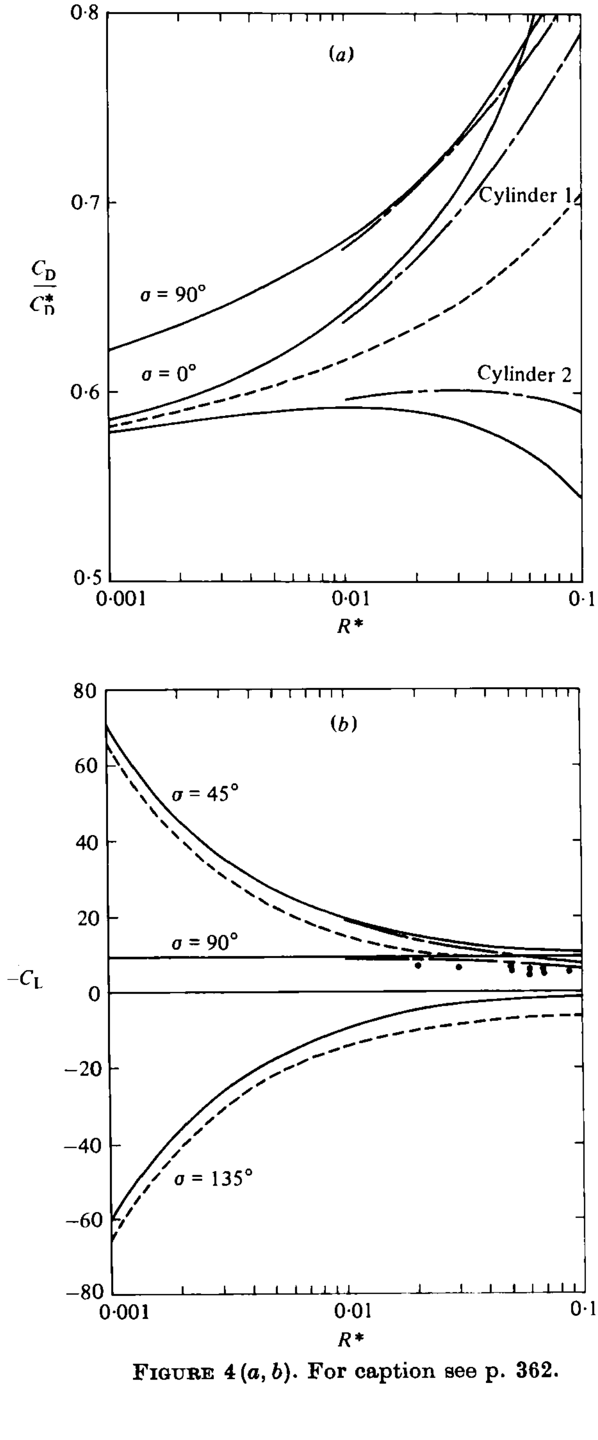

FIGURE 4 (a, b). For caption see p. 362.

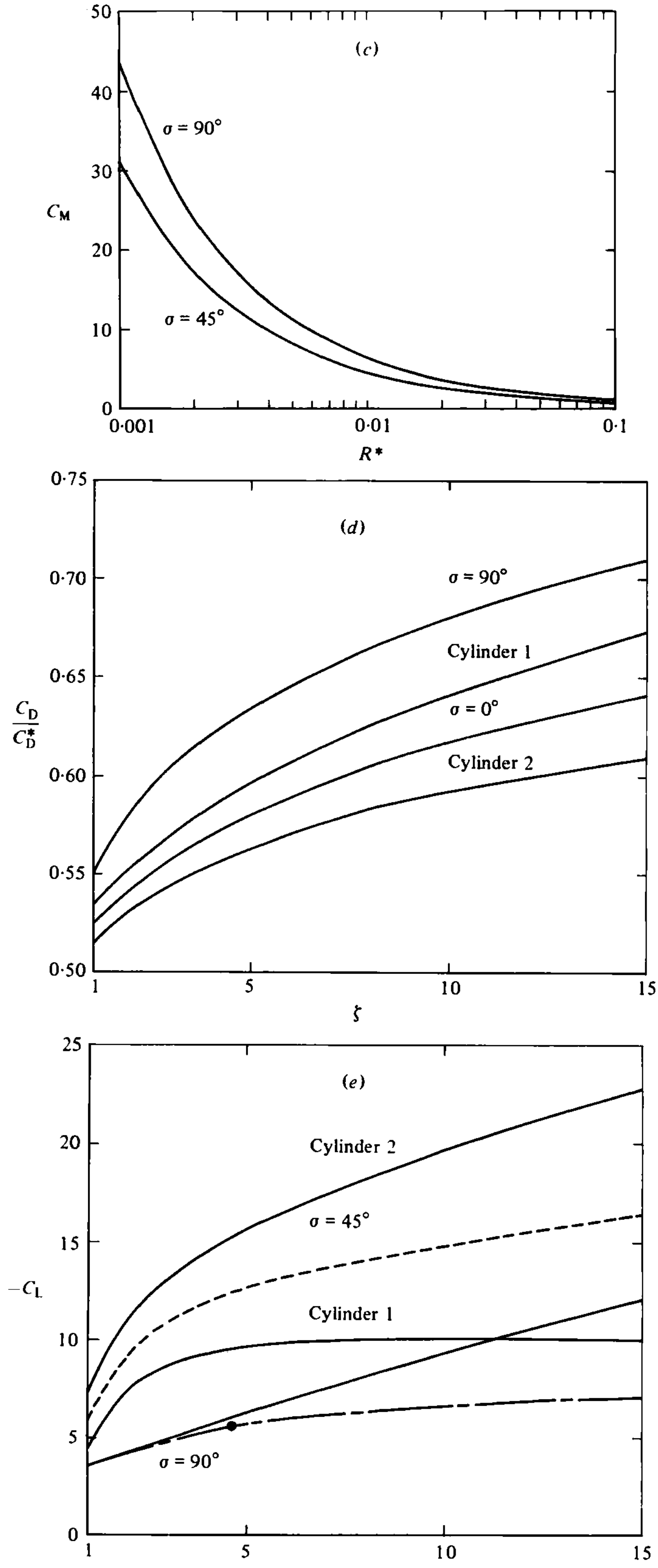

FIGURE 4 (*c–e*). For caption see p. 362.

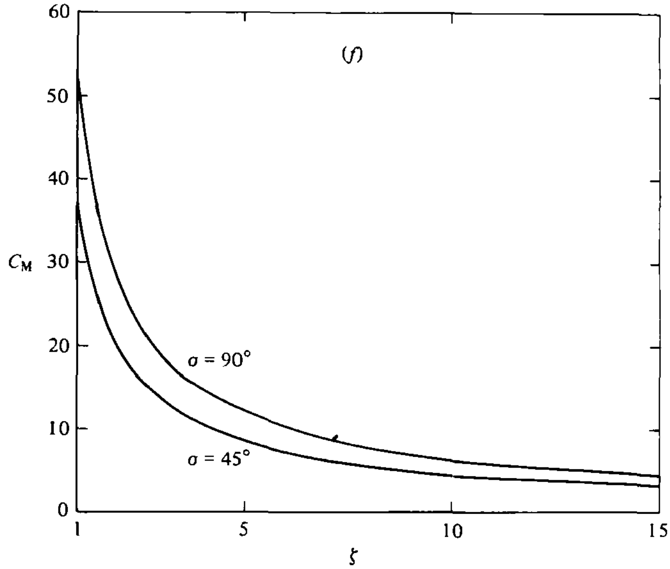

FIGURE 4. Parametric studies of the drag, lift and moment coefficients of the two equal circular cylinders. The solid curves for (40) plus (50) are to be compared with the dashed curves for (40) which contains no effects of fluid inertia. The dotted–broken curves are Yano & Kieda's (1980) results based on the Oseen equations, and the solid circles are the experimental data of Taneda (1957). (*a*) Variations of relative drag coefficients C_D/C_D^* with R^* for $\zeta = 10$ and $\sigma = 0°$ and 90°. (*b*) Variations of lift coefficients C_L of cylinder 1 with R^* for $\zeta = 10$ and $\sigma = 45°$, 90° and 135°. (*c*) Variations of moment coefficients C_M of cylinder 1 with R^* for $\zeta = 10$ and $\sigma = 45°$ and 90°. (*d*) Variations of relative drag coefficients C_D/C_D^* with ζ for $R^* = 0{\cdot}01$ and $\sigma = 0°$ and 90°. (*e*) Variations of lift coefficients C_L with ζ for $R^* = 0{\cdot}01$ and $\sigma = 45°$ and 90°. The solid curve for $\sigma = 90°$ is independent of R^*. The dashed curve is for $R^* = 0{\cdot}1$ and the solid circle is Taneda's datum for $R^* = 0{\cdot}015$. (*f*) Variations of moment coefficients C_M of cylinder 1 with ζ for $R^* = 0{\cdot}01$ and $\sigma = 45°$ and 90°.

8. Concluding remarks

Applying the method of matched asymptotic expansions we have analysed the low-Reynolds-number flow past two equal circular cylinders. The analytical expressions for the drag, lift and moment coefficients of the cylinders, (40) plus (50), could be successfully obtained by using bi-polar co-ordinates to describe the Stokes solutions in the inner expansions. Note, however, that there exists an upper limit of the parameter ζ or R^* above which our analysis is no longer valid. In the analysis each cylinder must be positioned so that the disturbed flow generated by the presence of the other cylinder in the uniform stream is governed by the Stokes equations. Thus we have the conditions

$$R = R^*\zeta \ll 1 \leqslant \zeta. \tag{61}$$

In figures 4(*a*–*g*) the graphs of the above hydrodynamic characteristics are plotted for comparison with other results. It is noticeable that the lift coefficient C_L for $\sigma = \frac{1}{2}\pi$ shown in figure 4(*b*) does not depend on R^*, and retains an intrinsic value as $R^* \to 0$. It also seems to be in good agreement with Taneda's (1957) experimental data.

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